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CONDITIONS FOR EXPONENTIAL ERGODICITY AND BOUNDS FOR THE DECAY PARAMETERS OF A BIRTH-DEATH PROCESS

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Conditions for exponential ergodicity and bounds for the decay parameters of a birth-death process *)

by

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ABSTRACT

This paper is addressed to two problems in connection with exponential ergodicity for birth-death processes on a semi-infinite lattice. The first is to determine from the birth and death rates whether exponential ergodicity prevails. We give some necessary and some sufficient conditions which suffice to settle the question for most processes encountered in practice. In particular, a complete solution is obtained for processes where, from some finite state n onwards, the associated rates are rational functions of n. The second, more difficult problem is to evaluate the decay parameter of an exponentially ergodic birth-death process. Our contribution to the solution of this problem consists of a number of upper and lower bounds.

KEY WORDS & PHRASES: *birth-death processes, dual birth-death processes, decay parameter, exponential ergodicity, orthogonal polynomials, spectral representation, transition probabilities*

*) This report has been submitted for publication elsewhere.

1. Introduction

Consider a standard, conservative Markov process in continuous time, whose state space $E \equiv \{0,1,2,\ldots\}$ constitutes an irreducible class. Let its (stationary) transition probabilities be denoted by $p_{ij}(t)$ $(i,j \in E,\ t \geq 0)$. The transition $i \to j$ is then said to be exponentially ergodic if $p_{ij}(t)$ tends to its ergodic limit p_j (independent of i because of the irreducibility of E) exponentially fast, i.e., if there exists an $\alpha > 0$ such that

$$(1.1) \qquad p_{ij}(t) - p_j = O(e^{-\alpha t})$$

as $t \to \infty$. We will study the phenomenon of exponential ergodicity in the context of birth-death processes and thus continue the works of Callaert (1971,1974) and Callaert and Keilson (1973a,1973b). Our main tool will be Karlin and McGregor's (1957a) spectral representation for the transition probabilities of a birth-death process, which says that for this type of Markov process

$$(1.2) \qquad p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) d\psi(x)$$

$(i,j \in E,\ t \geq 0)$. Here π_j are constants and $\{Q_n\}$ is a system of polynomials properly normalized and orthogonal with respect to the mass distribution $d\psi$, whose moments $\int x^n d\psi(x)$, $n = 0,1,\ldots$, are all finite and whose support (we shall also use the term spectrum) $S(d\psi)$, defined by

$$(1.3)\qquad S(d\psi) \equiv \{x \mid \int_{x-\varepsilon}^{x+\varepsilon} d\psi(\xi) > 0 \text{ for all } \varepsilon > 0\},$$

is infinite and contained in $[0,\infty)$.

In view of (1.2) the basic results of Kingman (1963a,1963b) on exponential ergodicity for Markov processes become transparent for birth-death processes. Take Kingman's (1963a) "solidarity" theorem for the transition probabilities of a transient or null-recurrent Markov process, which states that the maximal value of α in (1.1) is the same for each pair i,j, so that in particular either all or none of the $p_{ij}(t)$ go to their (zero) limits exponentially fast. (This common maximal value is then called the decay parameter of the process; if it is positive the process itself is called exponentially ergodic.) Now let $\gamma = \gamma(d\psi)$, where

$$(1.4)\qquad \gamma(d\psi) \equiv \inf \{x \mid x > 0 \text{ and } x \in S(d\psi)\},$$

and $d\psi$ the mass distribution associated with a transient or null-recurrent birth-death process. Then it is not difficult to see that for each pair i,j

$$(1.5)\qquad \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) d\psi(x) = O(e^{-\gamma t})$$

as $t \to \infty$. To show that for any pair i,j the factor γ in (1.5) cannot be improved (i.e., enlarged) is somewhat more troublesome. Callaert (1971,1974) uses a rather complicated argument involving theorems of Widder's on Stieltjes transforms and Laplace-Stieltjes transforms, but less sophisticated methods lead to the same conclusion. For it may be shown that $d\psi$ has no isolated point mass at 0 (Karlin and McGregor (1957b)), so that, actually, γ is the smallest point in $S(d\psi)$. A familiar theorem on zeros of orthogonal polynomials then implies that $Q_n(\gamma) \neq 0$ for all n. Subsequently using a straightforward argument of the type on p. 105 of Van Doorn (1981a) yields Callaert's result. Thus $\gamma(d\psi)$ is Kingman's decay parameter for a birth-death process with mass distribution $d\psi$ if the process is transient or null recurrent.

If a birth-death process is positive recurrent, then the associated mass distribution $d\psi$ has positive mass at 0. Indeed, we have (Karlin and McGregor (1957b))

$$(1.7) \qquad p_j = \pi_j d\psi(0) > 0$$

($j \in E$). Since the Q_n are normalized such that $Q_n(0) = 1$ for all n in this case, it follows that instead of (1.2) we can write

$$(1.8) \qquad p_{ij}(t) - p_j = \pi_j \int_{0+}^{\infty} e^{-xt} Q_i(x) Q_j(x) d\psi(x).$$

A small complication now arises, which is also reflected in Kingman's (1963b) result for positive recurrent Markov processes. For again we have

$$(1.9) \qquad \pi_j \int_{0+}^{\infty} e^{-xt} Q_i(x) Q_j(x) d\psi(x) = O(e^{-\gamma t})$$

as $t \to \infty$, where $\gamma = \gamma(d\psi)$, but there may be pairs i,j for which the factor γ in (1.9) can be improved. This contingency is brought about when $d\psi$ has an isolated point mass at γ and $Q_i(\gamma) = 0$ or $Q_j(\gamma) = 0$. This being an exceptional case (there is at most one n such that $Q_n(\gamma) = 0$), it is quite natural, indeed common practice, to call γ the decay parameter of the process and the process exponentially ergodic if $\gamma > 0$. Proofs for the above statements (which are Callaert's) may be given along the alternative lines sketched for the transient or null-recurrent case.

Summarizing, birth-death processes provide an illustrative example of Kingman's solidarity theorems for Markov processes in view of Karlin and McGregor's spectral representation (1.2) and Callaert's fundamental result (which can be given a relatively simple proof) that the decay parameter of a birth-death process equals $\gamma(d\psi)$, where $d\psi$ is the associated mass distribution.

Two obvious problems now arise in the context of birth-death processes, viz., (i) to give criteria for $\gamma(d\psi)$ to be positive in terms of the parameters which usually define a birth-death process (the birth and death rates), and more specifically (ii) to determine the value of $\gamma(d\psi)$ or at least bounds for $\gamma(d\psi)$ in terms of the rates. These are the problems to which this paper is adressed. As for (i) it will give us the opportunity to correct a statement in Van Doorn (1980) (unfortunately repeated in Van Doorn (1981a)), which was based on a misinterpretation of Callaert's results.

The plan of the paper is as follows. In Chapter 2 we will formally introduce the necessary concepts and results related to birth-death processes and, in particular, to the spectral representation for their transition probabilities. This formal introduction will incorporate a more general state space definition than given above, in that we allow absorbing states -1 and (implicitly) ∞. Of course, -1 and ∞ are quite different in character. While the inclusion of an absorbing state -1 is desirable in many applications, the inclusion of a reachable state ∞ is nót. The reason for allowing ∞ is that we do not wish to put a priori limitations on the values of the birth and death rates. It may then be necessary, however, to explain for disappearing probability mass, which is conveniently done by assigning it (implicitly) to a state ∞.

In Chapter 3 we give some useful characterizations for the decay parameter of a birth-death process. Then, in Chapter 4, we will obtain bounds on the decay parameter which are based on the foregoing characterizations. Most of the preparatory work in this respect is done in a separate paper (Van Doorn (1982)) which uses the more abstract terminology of orthogonal polynomials througout.

Problem (i) above will be tackled in Chapter 5. That is, we give conditions for a birth-death process to be exponentially ergodic. In particular we give the precise conditions for exponential ergodicity when, from some finite state n onwards, the birth and death rates are rational functions of n. Finally, we illustrate the results of the paper with some examples in Chapter 6.

2. Birth-death processes

2.1 Preliminaries

A birth-death process on the set $E' \equiv \{-1,0,1,\ldots\}$, where -1 is an absorbing barrier and $E \equiv \{0,1,\ldots\}$ constitutes an irreducible class, is faithfully represented by an array of functions $\{p_{ij}(t) \mid i,j \in E',\ t \geq 0\}$ (the transition probabilities), satisfying the conditions

$$\sum_j p_{ij}(t) \leq 1 \ , \tag{2.1}$$

$$p_{ij}(t) \geq 0 \ , \tag{2.2}$$

$$p_{ij}(0) = \delta_{ij} \ , \tag{2.3}$$

$$p_{ij}(t+s) = \sum_k p_{ik}(t)p_{kj}(s) \ , \tag{2.4}$$

$$p'_{ij}(t) = \sum_k a_{ik}p_{kj}(t) \ , \tag{2.5}$$

$$p'_{ij}(t) = \sum_k p_{ik}(t)a_{kj} \ , \tag{2.6}$$

for $i,j \in E'$ and $t,s \geq 0$. Here $a_{-1,j} = 0$ for all j, and, for $i \in E$,

$$a_{ij} = \begin{cases} \mu_i & \text{if } j = i-1 \\ -(\lambda_i + \mu_i) & \text{if } j = i \\ \lambda_i & \text{if } j = i+1 \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

where λ_i and μ_i, the <u>birth and death rates</u>, respectively, are positive with the exception $\mu_0 \geqq 0$. The <u>backward equations</u> (2.5) are equivalent to the more usual postulates

$$
\begin{aligned}
(2.8) \qquad & p_{i,i+1}(t) = \lambda_i t + o(t) \\
& p_{ii}(t) = 1 - (\lambda_i+\mu_i)t + o(t) \\
& p_{i,i-1}(t) = \mu_i t + o(t)
\end{aligned}
$$

as $t \to \infty$, for $i \in E$. The <u>forward equations</u> (2.6) are not always encountered as a postulate. However, it has been shown by Karlin and McGregor (1959) that these equations must be satisfied in order that the sample paths of the process are continuous except for simple discontinuities with saltus ± 1, which we consider a natural desideratum. The reason for allowing the $\leqq$ sign in (2.1) has been explained in the introduction. As is well known, any set $\mathcal{P} = \{\lambda_n, \mu_n\}_{n=0}^{\infty}$ of birth and death rates corresponds to at least one process $\{p_{ij}(t)\}$ satisfying (2.1)-(2.7).

The initial condition (2.3) and the forward equations (2.6) imply

$$
(2.9) \qquad p_{i,-1}(t) = \mu_0 \int_0^t p_{i0}(\tau)d\tau
$$

($i \in E$), while $p_{-1,j}(t) = \delta_{-1,j}$ for all j and $t \geqq 0$ by (2.3) and the backward equations (2.5). Otherwise the transition probabilities involving the absorbing state -1 do not enter in an essential way in (2.2)-(2.7). Therefore, we might as well forget about -1 and represent a birth-death process by an array of functions $\{p_{ij}(t) \mid i,j \in E,\ t \geqq 0\}$ satisfying (2.2)-(2.7) and

(2.10) $$\mu_0 \int_0^t p_{i0}(\tau)d\tau + \sum_j p_{ij}(t) \leqq 1$$

for $i,j \in E$ and $t,s \geqq 0$, where all summations extend over E instead of E'. This representation will be our starting point.

2.2 The spectral representation

Apart from some details which will be dealt with in Appendix 2, Karlin and McGregor (1957a) have proven the following fundamental result.

THEOREM 2.1. There exists a one-to-one correspondence between the set of arrays of functions $\{p_{ij}(t) \mid i,j \in E,\ t \geqq 0\}$ representing a birth-death process and the set of pairs $\{\mu_0, d\psi\}$ satisfying

(2.11) $\mu_0 \geqq 0$,

(2.12) $d\psi$ is a mass distribution on $[0,\infty)$ of total mass 1,

(2.13) $d\psi$ is extremal (see the remark below),

(2.14) $S(d\psi)$, the support of $d\psi$, is infinite,

(2.15) the moments $m_n \equiv \int_0^\infty x^n d\psi(x)$ are finite for all positive n,

(2.16) if $\mu_0 > 0$ then $\mu_0 \int_0^\infty d\psi(x)/x \leqq 1$.

For a pair $\{\mu_0, d\psi\}$ the corresponding functions $p_{ij}(t)$ are constructed as follows. Let $\{Q_n\}_{n=0}^\infty$ be the system of polynomials which is orthogonal with respect to $d\psi$ and normalized such that it satisfies a recurrence relation of the form

$$\lambda_n Q_{n+1}(x) = (\lambda_n + \mu_n - x) Q_n(x) - \mu_n Q_{n-1}(x) \quad , \ n > 0, \qquad (2.17)$$
$$\lambda_0 Q_1(x) = \lambda_0 + \mu_0 - x \ , \quad Q_0(x) = 1 \ .$$

Since μ_0 is given this uniquely determines the parameters λ_n and μ_n. Further, let $\{\pi_n\}_{n=0}^{\infty}$ be defined in terms of these parameters as

$$\pi_0 = 1 \ ; \quad \pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \quad , \ n > 0 \ . \qquad (2.18)$$

Then

$$p_{ij}(t) = \pi_j \int_0^{\infty} e^{-xt} Q_i(x) Q_j(x) d\psi(x) \qquad (2.19)$$

$(i,j \in E, \ t \geqq 0)$ and the λ_n and μ_n $(n \geqq 0)$ are the birth and death rates, respectively, of the process which is represented by $\{p_{ij}(t)\}$.

REMARK. A mass distribution $d\psi$ on $[0,\infty)$ is <u>extremal</u> if among all distributions $d\phi$ with the properties $\min S(d\phi) = \min S(d\psi) \equiv \xi$ and $\int x^n d\phi(x) = \int x^n d\psi(x) \equiv m_n$ for all nonnegative n, $d\psi$ has maximal mass at ξ (cf. Shohat and Tamarkin (1943), Theorem 2.13). Clearly, a distribution $d\psi$ is extremal when it is uniquely determined by its moments m_n, $n = 0,1,\ldots$. It will be useful to keep in mind that a sufficient condition for uniqueness (hence for extremality) of $d\psi$ is that its support be concentrated in a finite interval (cf. Chihara (1978), Theorem II.5.7).

Note that Theorem 2.1 does not specify how to construct the pair $\{\mu_0, d\psi\}$ for a birth-death process represented by a set $\{p_{ij}(t)\}$. Of course, μ_0 equals the death rate in the zero state and Q_n and π_n are uniquely determined by (2.17) and (2.18) if λ_n and μ_n are the birth and death rates of the process $\{p_{ij}(t)\}$. But the difficulty may arise that $d\psi$ is not uniquely determined by $\{Q_n\}$. This reflects the well-known fact that a birth-death process is not necessarily uniquely determined by its birth and death rates. If this situation prevails one needs an additional characterization to fix the process and thus the distribution $d\psi$. There are several ways for doing this. Karlin and McGregor (1957a) use a parameter π_∞ which can range continuously from 0 to ∞, inclusive, and which may be interpreted as a measure for the behaviour of a state ∞ which can be 'anything between' completely reflecting ($\pi_\infty = 0$) to completely absorbing ($\pi_\infty = \infty$). For our purposes it will be convenient (and possible) to choose as an extra characterization a parameter which can range continuously from some nonnegative value (zero if $\mu_0 = 0$, positive if $\mu_0 > 0$) to a larger, finite value, and which (with one exception) may be identified with the decay parameter of the pertinent process. These remarks will be substantiated later on.

2.3 The spectrum

We will investigate what information can be obtained on the spectrum (= support) of the mass distribution(s) associated with a set $\mathcal{P} = \{\lambda_n, \mu_n\}$ of birth and death rates. Our results will involve the parameters $\pi_n \equiv \pi_n(\mathcal{P})$ and polynomials $Q_n \equiv Q_n(\mathcal{P})$, which, as noted, are uniquely determined by $\mathcal{P}$ through (2.17) and (2.18). We first give some notation and preliminary results which can be found in Chihara (1978) (use (3.5) and (3.6)).

For all positive n, $Q_n(x)$ has n positive, distinct zeros $x_{n1}(\mathcal{P}) < x_{n2}(\mathcal{P}) < \ldots < x_{nn}(\mathcal{P})$ with the property

(2.20) $$x_{n+1,i}(\mathcal{P}) < x_{ni}(\mathcal{P}) < x_{n+1,i+1}(\mathcal{P})$$

(i = 1,2,...,n). Hence,

(2.21) $$\xi_i(\mathcal{P}) \equiv \lim_{n\to\infty} x_{ni}(\mathcal{P}) \quad \text{and} \quad \eta_j(\mathcal{P}) \equiv \lim_{n\to\infty} x_{n,n-j+1}(\mathcal{P})$$

(i,j = 1,2,...) exist (possibly $\eta_j(\mathcal{P}) = \infty$). Also,

(2.22) $$0 \leqq \xi_i(\mathcal{P}) \leqq \xi_{i+1}(\mathcal{P}) < \eta_{j+1}(\mathcal{P}) \leqq \eta_j(\mathcal{P}) \leqq \infty$$

(i,j = 1,2,...), so that both

(2.23) $$\sigma(\mathcal{P}) \equiv \lim_{i\to\infty} \xi_i(\mathcal{P}) \quad \text{and} \quad \tau(\mathcal{P}) \equiv \lim_{j\to\infty} \eta_j(\mathcal{P})$$

exist (possibly ∞). Furthermore,

(2.24) $$\xi_{i+1}(\mathcal{P}) = \xi_i(\mathcal{P}) \Rightarrow \sigma(\mathcal{P}) = \xi_i(\mathcal{P})$$

(i = 1,2,...) and

(2.25) $$\eta_{j+1}(\mathcal{P}) = \eta_j(\mathcal{P}) \Rightarrow \tau(\mathcal{P}) = \eta_j(\mathcal{P})$$

(j = 0,1,...), where $\eta_0(\mathcal{P}) \equiv \infty$. From (2.25) and Chihara (1978), Theorems IV.3.1 and IV.3.3 we readily obtain

(2.26) $$\tau(\mathcal{P}) < \infty \iff \eta_1(\mathcal{P}) < \infty \iff \sup \mathcal{P} < \infty .$$

We finally define

(2.27) $$\Xi(\mathcal{P}) \equiv \{\xi_1(\mathcal{P}),\xi_2(\mathcal{P}),...\}$$

and, if $\sup \mathcal{P} < \infty$,

(2.28) $$H(\mathcal{P}) \equiv \{\eta_1(\mathcal{P}),\eta_2(\mathcal{P}),...\}.$$

Evidently, both sets may be finite.

<u>THEOREM 2.2.</u> For a set $\mathcal{P} = \{\lambda_n,\mu_n\}$ of birth and death rates and associated polynomials Q_n the following hold:

(i) If the series

(2.29) $$\sum_{n=0}^{\infty}(\pi_n + (\lambda_n\pi_n)^{-1})$$

diverges, then there is exactly one distribution $d\psi$ satisfying (2.12) - (2.16) with respect to which the polynomials Q_n are orthogonal (and therefore a unique birth-death process with rate set $\mathcal{P}$). If $\sigma(\mathcal{P}) = \infty$, then

$$(2.30) \qquad S(d\psi) = \Xi(\mathcal{P}).$$

If $\sigma(\mathcal{P}) < \infty$ and $\sup \mathcal{P} = \infty$, then

$$(2.31) \qquad S(d\psi) = \overline{\Xi}(\mathcal{P}) \cup S_1(\mathcal{P})$$

(a bar denoting closure) and $S_1(\mathcal{P}) \subset (\sigma(\mathcal{P}),\infty)$ (possibly, $S_1(\mathcal{P}) = \emptyset$); also $\sigma(\mathcal{P})$ is the smallest limit point of $S(d\psi)$. Finally, if $\sigma(\mathcal{P}) < \infty$ and $\sup \mathcal{P} < \infty$, then

$$(2.32) \qquad S(d\psi) = \overline{\Xi}(\mathcal{P}) \cup S_1(\mathcal{P}) \cup \overline{H}(\mathcal{P}),$$

and $S_1(\mathcal{P}) \subset (\sigma(\mathcal{P}),\tau(\mathcal{P}))$ (possibly, e.g., if $\sigma(\mathcal{P}) = \tau(\mathcal{P})$, $S_1(\mathcal{P}) = \emptyset$); also, $\sigma(\mathcal{P})$ ($\tau(\mathcal{P})$) is the smallest (largest) limit point of $S(d\psi)$.

(ii) If the series (2.29) converges, then there is an infinite number of distributions satisfying (2.12) - (2.16) with respect to which the polynomials Q_n are orthogonal (and therefore an infinite number of birth-death processes with rate set $\mathcal{P}$). Each of these distributions has a discrete spectrum with no finite limit point. The spectral points of such a distribution separate those of any other solution. One has $\xi_1(\mathcal{P}) > 0$ and each solution is determined by the single spectral point ξ

which is smaller than or equal to $\xi_1(\mathcal{P})$. If $\mu_0 = 0$, then ξ ranges continuously from 0 to $\xi_1(\mathcal{P})$ inclusive, while for $\mu_0 > 0$, ξ ranges continuously from some value $\omega(\mathcal{P})$, $0 < \omega(\mathcal{P}) < \xi_1(\mathcal{P})$, to $\xi_1(\mathcal{P})$ inclusive. Denoting by $d\psi_\xi$ the solution whose smallest spectral point is ξ, $\omega(\mathcal{P})$ is the unique solution of the equation $\phi(\xi) = 1$, where

$$(2.33) \qquad \phi(\xi) \equiv \mu_0 \int_0^\infty d\psi_\xi(x)/x \ .$$

Finally,

$$(2.34) \qquad S(d\psi_{\xi_1(\mathcal{P})}) = \Xi(\mathcal{P}) \ .$$

PROOF. The problem of finding a distribution $d\psi$ with respect to which the polynomials Q_n are orthogonal and which satisfies (2.12), (2.14) and (2.15) may be formulated as a Stieltjes moment problem (Karlin and McGregor (1957a)). If this moment problem has exactly one solution then the statements in (i) concerning the spectrum of this solution are known (see Chihara (1978), Section II.4). If the Stieltjes moment problem has more than one solution, then the statements in (ii) with the exception of the lower bound on ξ for $\mu_0 > 0$, apply to the set of solutions which satisfy the extremality condition (2.13), as appears from Shohat and Tamarkin (1943), Theorem 2.13,and Chihara (1968).

Now if $\mu_0 = 0$, then, by Karlin and McGregor (1957a), Theorem 14, the divergence of (2.29) is equivalent to uniqueness of the solution of the Stieltjes moment problem. So we have dealt with this case.

If $\mu_0 > 0$, then, by Karlin and McGregor (1957a), Theorem 15, the divergence of (2.29) is equivalent to either uniqueness of the solution of the moment problem or $\phi(\xi_1(\mathcal{P})) = 1$. In the former case the validity of (i) has been established, therefore suppose that the moment problem has more than one (extremal) solution. Then, as is shown in Appendix 3, $\phi(\xi)$ is decreasing from $+\infty$ at $\xi = 0$ to some value $\lneqq 1$ at $\xi = \xi_1(\mathcal{P})$. Hence, if $\phi(\xi_1(\mathcal{P})) = 1$ then there is exactly one solution satisfying (2.12) - (2.15) and (2.16), which settles (i) for $\mu_0 > 0$ (apparently, this solution has an infinite spectrum with no finite limit point). Finally, if $\phi(\xi_1(\mathcal{P})) < 1$ and the moment problem has more than one solution, or equivalently, if (2.29) converges, then the lower bound $\omega(\mathcal{P})$ for the first spectral point of a distribution satisfying (2.12) - (2.16) is a direct consequence of the aforementioned behaviour of $\phi(\xi)$. □

If for a set of rates $\mathcal{P} = \{\lambda_n, \mu_n\}$ the series (2.29) converges, or equivalently, if $\mathcal{P}$ does not uniquely determine an associated distribution , or equivalently, if $\mathcal{P}$ does not uniquely determine an associated birth-death process , we write $p_{ij}(t,\xi)$ for the transition probabilities of the birth-death process associated with $d\psi_\xi$, where $d\psi_\xi$ has the interpretation given in the above theorem and ξ can range in the intervals indicated there. Karlin and McGregor (1957a) show that in this case

(2.35) $$p_{ij}(t,\xi_b) \leqq p_{ij}(t,\xi_a)$$

$(i,j \in E,\ t \geqq 0)$ if $\xi_a < \xi_b$. It is therefore natural to call $\{p_{ij}(t,\xi)\}$ the <u>minimal process</u> if $\xi = \xi_1(\mathcal{P})$ (it corresponds to what is often called the 'minimal Feller process'), and the <u>maximal process</u> if $\xi = 0$ $(\mu_0 = 0)$ or $\xi = \omega(\mathcal{P})$ $(\mu_0 > 0)$. Also, in the case of non-uniqueness we will write

(2.36) $$d\psi_{min} \equiv d\psi_{\xi_1(\mathcal{P})}$$

and

(2.37) $$d\psi_{max} = \begin{cases} d\psi_0 & \text{if } \mu_0 = 0 \\ d\psi_{\omega(\mathcal{P})} & \text{if } \mu_0 > 0 \end{cases}$$

(this definition of $d\psi_{max}$ differs from that of Karlin and McGregor (1957a)). Note that for $\mu_0 > 0$

(2.38) $$\mu_0 \int_0^\infty d\psi_{max}(x)/x = 1 \ .$$

We are mainly interested in minimal processes, but it will appear in the next section that maximal processes play an important role in the analysis of minimal processes.

We conclude this section with a lemma concerning the mass of a distribution at its spectral points, to which we will have reference later on. The result is given in Corollary 2.6 and Theorem 2.13 of Shohat and Tamarkin (1943).

<u>LEMMA 2.3.</u> If the polynomials Q_n associated with a set $\mathcal{P}$ =

$\{\lambda_n,\mu_n\}$ of birth and death rates are orthogonal with respect to a unique distribution $d\psi$ satisfying (2.12), (2.14) and (2.15) then

$$(2.39) \qquad d\psi(x) = \left[\sum_{n=0}^{\infty} \pi_n Q_n^2(x)\right]^{-1}$$

for all x, which is interpreted as zero if the sum diverges. If the polynomials do not uniquely determine a distribution, then the right hand side of (2.39) is positive for all x and equals the maximal mass any distribution associated with the Q_n can have at x. If for some distribution $d\psi$ and some x (2.39) holds, then $d\psi$ is an extremal distribution.

2.4 Duality

As a final and essential prerequisite we must introduce a duality concept for birth-death processes which was implicitly mentioned in Karlin and McGregor (1957a) and explicitly in Karlin and McGregor (1957b). For a set $\mathcal{P} = \{\lambda_n,\mu_n\}$ of birth and death rates, the dual set $\mathcal{P}^d = \{\lambda_n^d,\mu_n^d\}$ is defined by

$$(2.40) \qquad \begin{array}{l} \mu_0 = 0 \implies \lambda_n^d = \mu_{n+1} \, , \quad \mu_n^d = \lambda_n \\ \mu_0 > 0 \implies \mu_0^d = 0 \, , \quad \lambda_n^d = \mu_n \, , \quad \mu_{n+1}^d = \lambda_n \end{array}$$

$(n = 0,1,\dots)$. Clearly, this duality concept establishes a one-to-one correspondence between the sets of rates where $\mu_0 = 0$ and those where $\mu_0 > 0$. The relations between the sets of polynomials corresponding to $\mathcal{P}$ and $\mathcal{P}^d$ are readily found to be

$$(2.41)\qquad \begin{aligned} \mu_0 = 0 &\Rightarrow Q_n^d(x) = \lambda_n\pi_n(Q_{n+1}(x)-Q_n(x))/(-x) \\ \mu_0 > 0 &\Rightarrow Q_{n+1}^d(x) = \lambda_n\pi_n(Q_{n+1}(x)-Q_n(x))/\mu_0 \end{aligned}$$

$(n = 0,1,\ldots)$. For $\mu_0 = 0$, the polynomials Q_n^d are known as the kernel polynomials with parameter 0 corresponding to the polynomials Q_n (cf. Chihara (1978)). There exists a separation theorem for the zeros of kernel polynomials (Chihara (1978), Theorem I.7.2) which, in view of (2.21), is easily seen to lead to

$$(2.42)\qquad \begin{aligned} \mu_0 = 0 &\Rightarrow \xi_i(\mathcal{P}) \leqq \xi_i(\mathcal{P}^d) \leqq \xi_{i+1}(\mathcal{P}) \\ \mu_0 > 0 &\Rightarrow \xi_i(\mathcal{P}^d) \leqq \xi_i(\mathcal{P}) \leqq \xi_{i+1}(\mathcal{P}^d) \end{aligned}$$

$(i = 1,2,\ldots)$.

Regarding the parameters π_n and π_n^d associated with $\mathcal{P}$ and $\mathcal{P}^d$, respectively, we clearly have

$$(2.43)\qquad \begin{aligned} \mu_0 = 0 &\Rightarrow \pi_n^d = \lambda_0(\lambda_n\pi_n)^{-1} \\ \mu_0 > 0 &\Rightarrow \pi_n^d = \mu_0(\mu_n\pi_n)^{-1} \end{aligned}$$

$(n = 0,1,\ldots)$. Since $\lambda_n\pi_n = \mu_{n+1}\pi_{n+1}$, it follows that

$$(2.44)\qquad \sum_{n=0}^{\infty}(\pi_n + (\lambda_n\pi_n)^{-1}) = \infty \iff \sum_{n=0}^{\infty}(\pi_n^d + (\lambda_n^d\pi_n^d)^{-1}) = \infty.$$

The duality relations between the mass distributions associated with $\mathcal{P}$ and $\mathcal{P}^d$ are somewhat more intricate because of the non-uniqueness that may occur. However, the following theorem shows that in that event certain relations exist between the distributions associated with maximal and minimal processes, which is all we need.

<u>THEOREM 2.4</u> (i). If (2.29) diverges, so that $d\psi$ is uniquely determined by $\mathcal{P}$, then the dual distribution $d\psi^d$ is uniquely determined by $\mathcal{P}^d$ and given by

$$(2.45)\qquad \begin{aligned} \mu_0 = 0 &\Rightarrow d\psi^d(x) = x\,d\psi(x)/\lambda_0 \qquad x \geq 0 \\ \mu_0 > 0 &\Rightarrow d\psi^d(x) = \begin{cases} 1 - \mu_0 \int_0^\infty d\psi(x)/x & \text{if } x = 0 \\ \mu_0 d\psi(x)/x & \text{if } x > 0. \end{cases} \end{aligned}$$

(ii). If (2.29) converges, so that the distribution associated with $\mathcal{P}$ is not uniquely determined, then the distribution associated with $\mathcal{P}^d$ is not uniquely determined either, but one has

$$(2.46)\qquad \begin{aligned} \mu_0 = 0 &\Rightarrow d\psi^d_{min}(x) = x\,d\psi_{max}(x)/\lambda_0 \qquad x \geq 0 \\ \mu_0 > 0 &\Rightarrow d\psi^d_{min}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \mu_0 d\psi_{max}(x)/x & \text{if } x > 0 \end{cases} \end{aligned}$$

and

$$(2.47)\quad \begin{cases} \mu_0 = 0 \implies d\psi^d_{max}(x) = x\,d\psi_{min}(x)/\lambda_0 & x \geq 0 \\ \mu_0 > 0 \implies d\psi^d_{max}(x) = \begin{cases} 1 - \mu_0 \int_0^\infty d\psi_{min}(x)/x & \text{if } x = 0 \\ \mu_0\, d\psi_{min}(x)/x & \text{if } x > 0. \end{cases} \end{cases}$$

PROOF. (i) If (2.29) diverges, then, by (2.44), Theorem 2.1 and Karlin and McGregor (1957a), Theorems 14 and 15, the distributions $d\psi$ and $d\psi^d$ are uniquely determined by $\mathcal{P}$ and $\mathcal{P}^d$, respectively. By Lemmas 3 and 2 of Karlin and McGregor (1957a) the right hand sides in (2.45) are distributions which satisfy the conditions (2.12) and (2.14) - (2.16) and with respect to which the polynomials Q_n and Q_n^d are orthogonal. Finally, from Karlin and McGregor (1957a), Theorems 9, 14 and 15 it is seen that the extremality condition (2.13) is redundant when (2.29) diverges, so that (2.45) holds.

The proof of (ii) has been relegated to Appendix 4. □

3. Representations for the decay parameter

Consider a birth-death process $\{p_{ij}(t) \mid i,j \in E,\ t \geq 0\}$ satisfying (2.2) - (2.7) and (2.10). Then, as is well known, $p_{ij}(t)$ tends to a limit p_j as $t \to \infty$. The decay parameter α^* for this process is defined as

$$(3.1) \quad \alpha^* \equiv \sup \{\alpha \geq 0 \mid p_{ij}(t) - p_j = O(e^{-\alpha t}) \text{ as } t \to \infty \text{ for all } i,j \in E\},$$

and the process is said to be exponentially ergodic if $\alpha^* > 0$.

By Theorem 2.1 the process $\{p_{ij}(t)\}$ can be represented by a pair $\{\mu_0, d\psi\}$ satisfying the conditions (2.11) - (2.16). With $\gamma(d\psi)$ defined as in (1.4) we now have the following.

THEOREM 3.1 (Callaert (1971,1974)). The decay parameter of a birth-death process represented by $\{\mu_0, d\psi\}$ equals $\gamma(d\psi)$.

Callaert has formulated this theorem in a somewhat less general context in that he assumes (implicitly in Callaert (1974)) $d\psi$ to be uniquely determined by the birth and death rates. It is easily seen, however, that this assumption is essential neither in his original proof nor in the alternative proof sketched in the introduction (where we have assumed $\mu_0 = 0$, but, again, this is no essential restriction).

REMARK. Using the result (A.15) of Appendix 2 it can readily be verified that $\gamma(d\psi)$ is also precisely the decay parameter for $p_{i,-1}(t)$ $(i \in E)$.

Our next step will be to relate $\gamma(d\psi)$ to the points $\xi_i(\mathcal{P})$ of (2.21), where $\mathcal{P} = \{\lambda_n,\mu_n\}$ is the unique set of birth and death rates associated with the pair $\{\mu_0,d\psi\}$. As a preliminary result we state the following lemma, which should replace the invalid statements (5.10) - (5.12) and (5.16) - (5.17) in Van Doorn (1980) alluded to earlier (without affecting the main results of that paper).

LEMMA 3.2. For any set $\mathcal{P} = \{\lambda_n,\mu_n\}$ of birth and death rates one has

(i) $\mu_0 = 0$ and $\sum (\lambda_n\pi_n)^{-1} = \infty \Rightarrow \xi_1(\mathcal{P}) = 0$,

(ii) $\mu_0 = 0$ and $\sum (\lambda_n\pi_n)^{-1} < \infty \Rightarrow \xi_1(\mathcal{P}) > 0$ or $\sigma(\mathcal{P}) = 0$

(iii) $\mu_0 > 0 \Rightarrow \xi_1(\mathcal{P}) > 0$ or $\sigma(\mathcal{P}) = 0$

(iv) $\sum \pi_n = \sum (\lambda_n\pi_n)^{-1} = \infty \Rightarrow \sigma(\mathcal{P}) = 0$,

(v) $\sum \pi_n < \infty$ and $\sum (\lambda_n\pi_n)^{-1} < \infty \Rightarrow \xi_1(\mathcal{P}) > 0$.

PROOF. If $\mu_0 = 0$ and $\sum(\lambda_n\pi_n)^{-1} = \infty$, then, by Lemma 6 of Karlin and McGregor (1957a), the (unique) distribution associated with $\mathcal{P}$ has an infinite moment of order -1, so that for each $\varepsilon > 0$, it has positive mass in the interval $[0,\varepsilon)$. It follows by Theorem 2.2 (i) that $\xi_1(\mathcal{P}) = 0$, which settles (i). If, in addition, $\sum\pi_n = \infty$, then, by Lemma 2.3 (note that $Q_n(0) = 1$ if $\mu_0 = 0$), the distribution has no mass at 0, so that 0 must be a limit point of its spectrum. Hence, by Theorem 2.2, $\sigma(\mathcal{P}) = 0$.

This proves (iv) for $\mu_0 = 0$. By (2.44) and (2.45) the same conclusion is easily seen to be valid for $\mu_0 > 0$ as well.

Statement (v) is contained in Theorem 2.2 (ii). So in proving (ii) we can assume that $\sum\pi_n = \infty$, implying that $\mathcal{P}$ uniquely determines a distribution $d\psi$. From Lemma 6 of Karlin and McGregor (1957a) we can then conclude that $d\psi$ has a finite moment of order -1, so that $d\psi$ has no mass at 0. The required result follows immediately.

Finally, if $\mu_0 > 0$, then, by (2.16), $d\psi$ has a finite moment of order -1, whence (iii) follows. □

Assuming for the moment that (2.29) diverges for a set $\mathcal{P} = \{\lambda_n, \mu_n\}$, so that $d\psi$ is uniquely determined by $\mathcal{P}$, it is evident from Theorem 2.2 (i) that $\gamma(d\psi) = \xi_1(\mathcal{P})$ if $\xi_1(\mathcal{P}) > 0$ and $\gamma(d\psi) = \xi_2(\mathcal{P})$ if $\xi_1(\mathcal{P}) = 0$ (if $\xi_2(\mathcal{P}) = 0$ too, then, by (2.24), $\sigma(\mathcal{P}) = 0$, so that $\gamma(d\psi) = 0$, since $\sigma(\mathcal{P})$ is a limit point of $S(d\psi)$). Although characterizations for $\xi_2(\mathcal{P})$ can be given, it is much easier to work with $\xi_1(\mathcal{P})$. This consideration leads us to bringing dual processes in our analysis, which has the additional advantage that the decay parameters of certain processes where $d\psi$ is not uniquely determined by $\mathcal{P}$ can be identified. The precise results are given in the next theorem.

<u>THEOREM 3.3</u>. Let $\{\mu_0, d\psi\}$ represent a birth-death process with rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$.

(i) If $\sum(\pi_n + (\lambda_n\pi_n)^{-1}) = \infty$, then

(3.2) $$\gamma(d\psi) = \begin{cases} \xi_1(\mathcal{P}) & \text{if } \mu_0 > 0 \\ \xi_1(\mathcal{P}^d) & \text{if } \mu_0 = 0 . \end{cases}$$

(ii) If $\sum(\pi_n + (\lambda_n\pi_n)^{-1}) < \infty$, then

(3.3) $$\mu_0 > 0 \implies$$
$$0 < \xi_1(\mathcal{P}^d) = \gamma(d\psi_{max}) \leqq \gamma(d\psi) \leqq \gamma(d\psi_{min}) = \xi_1(\mathcal{P})$$

(3.4) $$\mu_0 = 0 \implies$$
$$0 < \gamma(d\psi) \leqq \gamma(d\psi_{min}) = \xi_1(\mathcal{P}) \text{ or } \gamma(d\psi) = \gamma(d\psi_{max}) = \xi_1(\mathcal{P}^d)$$

Moreover, $d\psi$ is uniquely determined by $\mathcal{P}$ and the value of $\gamma(d\psi)$.

PROOF. Follows readily from Theorems 2.2 and 2.4 and Lemma 3.2. □

Of course, if for a set $\mathcal{P} = \{\lambda_n, \mu_n\}$ which uniquely determines the associated distribution $d\psi$, one knows beforehand that $\xi_1(\mathcal{P}) = \xi_1(\mathcal{P}^d)$ (as in the cases (ii), (iii) and (iv) of Lemma 3.2) one simply has $\gamma(d\psi) = \xi_1(\mathcal{P})$ and there is no need to consider the dual process. But, clearly, one of the attractions of Theorem 3.3 (i) is that such preconsiderations are not required; one has at one's disposal a simple criterion for choosing a rate set and associated distribution whose first spectral point can be identified with the decay parameter of the process under consideration.

The problem of finding bounds on the decay parameter of a birth-death process can now be formulated as the problem of finding bounds on $\xi_1(\mathcal{P})$ for a set $\mathcal{P} = \{\lambda_n,\mu_n\}$ (assuming we are only interested in the minimal (or maximal) process in the case of non-uniqueness). Before tackling this problem (in the next chapter), we will give some characterizations for $\xi_1(\mathcal{P})$. To this end we note that by defining

(3.5) $$P_n(x) = (-1)^n \lambda_0\lambda_1\cdots\lambda_{n-1}Q_n(x)$$

($n = 1,2,\ldots$), where the Q_n are the polynomials associated with $\mathcal{P}$, (2.17) can be written as

(3.6) $$\begin{aligned} P_{n+1}(x) &= (x-\lambda_n-\mu_n)P_n(x) - \lambda_{n-1}\mu_n P_{n-1}(x), \quad n > 0 \\ P_1(x) &= x - \lambda_0 - \mu_0; \; P_0(x) = 1 . \end{aligned}$$

With this identification we obtain from Van Doorn (1982) the following theorem, where $S(\underset{\sim}{Q}(x))$ denotes the number of sign changes in the sequence

(3.7) $$\underset{\sim}{Q}(x) = (Q_0(x),Q_1(x),Q_2(x),\ldots)$$

by deleting all zero terms.

<u>THEOREM 3.4.</u> For a set $\mathcal{P} = \{\lambda_n,\mu_n\}$ of birth and death rates one has

(3.8) $$\xi_1(\mathcal{P}) = \sup\{x \mid S(\underset{\sim}{Q}(x)) = 0\} .$$

The second characterization for $\xi_1(\mathcal{P})$ stems from the work of Chihara. We must give some definitions first.

A sequence $\{\beta_n\}_{n=1}^{\infty}$ is called a chain sequence if there exists a sequence $\{g_k\}_{k=0}^{\infty}$ with $0 \leqq g_0 < 1$ and $0 < g_k < 1$ $(k > 0)$, such that $\beta_n = (1-g_{n-1})g_n$; $\{g_k\}$ is called a parameter sequence for $\{\beta_n\}$. For instance, $\{\frac{1}{4}\}$ is a chain sequence for which $\{\frac{1}{2}\}$ is a parameter sequence. Now let

$$\alpha_n(x) \equiv \frac{\lambda_{n-1}\mu_n}{(\lambda_n+\mu_n-x)(\lambda_{n-1}+\mu_{n-1}-x)} \tag{3.9}$$

(x real, $n \geqq 1$). From Chihara (1978), Theorem IV.2.1 we then obtain the following.

THEOREM 3.5. For any rate set $\mathcal{P} = \{\lambda_n,\mu_n\}$ one has

$$\xi_1(\mathcal{P}) = \sup\{x \mid x < \lambda_n + \mu_n \text{ for all } n, \text{ and } \{\alpha_n(x)\} \text{ is a chain sequence}\}. \tag{3.10}$$

Other characterizations for $\xi_1(\mathcal{P})$ are possible (e.g., in terms of continued fractions), but Theorems 3.4 and 3.5 are basic to the bounds given in the next chapter.

4. Bounds on the decay parameter

We shall assume in this chapter that one is interested in a decay parameter which equals the limit point $\xi_1(\mathcal{P})$ of the smallest zeros $x_{n1}(\mathcal{P})$ of the polynomials Q_n, $n = 1,2,\ldots,$ associated with a set $\mathcal{P} = \{\lambda_n,\mu_n\}$ of birth and death rates. Whether these are the rates of the process under consideration or its dual process must be decided from Theorem 3.3. Henceforth, we will use the convention $\lambda_{-1} \equiv 0$.

4.1 Lower bounds

With (3.5) and (3.6) we obtain from Van Doorn (1982), Theorem 7, a lower bound for $\xi_1(\mathcal{P})$ which is based on the identification (3.8).

THEOREM 4.1. For any sequence $\{\chi_n\}_{n=0}^{\infty}$ of positive numbers one has

$$(4.1) \qquad \xi_1(\mathcal{P}) \geq \inf_{n\geq 0} \{\lambda_n + \mu_n - \frac{\lambda_{n-1}\mu_n}{\chi_n} - \chi_{n+1}\} .$$

The bound in (4.1) is best possible in the sense that numbers χ_n exist such that equality holds in (4.1). However, the problem of determining these χ_n is at least as difficult as the problem of finding $\xi_1(\mathcal{P})$ (see Van Doorn (1982)).

A simple lower bound is obtained by taking

$$(4.2) \qquad \chi_n = (\lambda_{n-1}\mu_n)^{\frac{1}{2}}$$

($n = 1,2,\ldots$) and χ_0 arbitrary. We then have

COROLLARY 4.2. $\xi_1(\mathcal{P}) \geq \inf_{n \geq 0} \{\lambda_n + \mu_n - (\lambda_{n-1}\mu_n)^{\frac{1}{2}} - (\lambda_n\mu_{n+1})^{\frac{1}{2}}\}$.

Another possibility is to let

$$(4.3) \qquad \chi_n = \lambda_{n-1}\mu_n/(\lambda_n+\mu_n-\theta)$$

($n = 1,2,\ldots$), where $\theta < \lambda_n + \mu_n$ for all $n > 0$. This is easily seen to yield a result which is essentially due to Léopold (1982).

COROLLARY 4.3. $\xi_1(\mathcal{P}) \geq \sup_{\theta\in\Theta} \inf_{n\geq 1} \{\theta - \frac{\lambda_{n-1}\mu_n}{\lambda_n + \mu_n - \theta}\}$,

where $\Theta \equiv \{\theta \mid \theta \leq \lambda_0 + \mu_0$ and $\theta < \lambda_n + \mu_n$ for all $n > 0\}$.

Chihara's characterization (3.10) may be shown to lead to the next lower bound (see Chihara (1978), Theorem IV.3.3 for the type of argument required, and Van Doorn (1982)).

THEOREM 4.4. For any chain sequence $\{\beta_n\}_{n=1}^{\infty}$ one has

$$(4.4) \qquad 2\xi_1(\mathcal{P}) \geq \inf_{n\geq 0} \{\nu_n + \nu_{n+1} - ((\nu_{n+1}-\nu_n)^2 + 4\lambda_n\mu_{n+1}/\beta_{n+1})^{\frac{1}{2}}\}$$

where $\nu_n = \lambda_n + \mu_n$.

Again, (4.4) is best possible, but there exists no convenient expression for the β_n which yield equality in (4.4). The most obvious choice would be $\{\beta_n\} = \{\frac{1}{4}\}$, whence we obtain

COROLLARY 4.5. $2\xi_1(\mathcal{P}) \geq \inf_{n\geq 0} \{\nu_n + \nu_{n+1} - ((\nu_{n+1}-\nu_n)^2 + 16\lambda_n\mu_{n+1})^{\frac{1}{2}}\}$

A slightly better result may be obtained by taking

$$\beta_n = \tfrac{1}{4} + \frac{1}{16n(n+1)} \tag{4.5}$$

(cf. Chihara (1978), p. 98). Of course, depending on the birth and death rates, it may very well be that even better results can be obtained for chain sequences which are not monotone such as $\{\frac{1}{9},\frac{4}{9},\frac{1}{9},\frac{4}{9},\frac{1}{9},\ldots\}$.

4.2 Upper bounds

Using (3.5) and (3.6), Theorem 4 of Van Doorn (1982) yields the following upper bound for $\xi_1(\mathcal{P})$, where, as in Theorem 4.4, $\nu_n \equiv \lambda_n + \mu_n$.

THEOREM 4.6. For any sequence $\{\chi_n\}_{n=0}^{\infty}$ of positive numbers and integers $k \geq 0$ and $M > 0$ one has

$$\xi_1(\mathcal{P}) < \left(\frac{\nu_M}{\chi_M} + \sum_{m=M+1}^{M+k}\left(\frac{\nu_m}{\chi_m} - 2\left(\frac{\lambda_{m-1}\mu_m}{\chi_{m-1}\chi_m}\right)^{\frac{1}{2}}\right)\right)\left(\sum_{m=M}^{M+k}\frac{1}{\chi_m}\right)^{-1} \tag{4.6}$$

Taking $k = 0$ and $\chi_n = 1$ for all n, Theorem 4.5 yields

$$\xi_1(\mathcal{P}) < \lambda_n + \mu_n \tag{4.7}$$

$(n = 0,1,\ldots)$. Letting $k = 1$ and $\chi_n = 1$ for all n gives us

$$\xi_1(\mathcal{P}) < \tfrac{1}{2}(\nu_n + \nu_{n+1}) - (\lambda_n \mu_{n+1})^{\frac{1}{2}} \tag{4.8}$$

$(n = 0,1,\ldots)$, which is not necessarily a better bound than (4.7). However, a simple argument (see Van Doorn (1982)) leads to a bound which improves upon both (4.7) and (4.8).

COROLLARY 4.7. $\xi_1(\mathcal{P}) < \tfrac{1}{2}(\nu_n + \nu_{n+1} - ((\nu_{n+1}-\nu_n)^2 + 4\lambda_n\mu_{n+1})^{\frac{1}{2}})$,

$n = 0,1,\ldots$.

5. Conditions for exponential ergodicity

5.1 Necessary and sufficient conditions

Let $\{\mu_0, d\psi\}$ represent a birth-death process with rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$. By Theorem 3.1 this process is exponentially ergodic if and only if $\gamma(d\psi) > 0$. An equivalent condition in terms of $\mathcal{P}$ is given in the following simple but useful theorem.

THEOREM 5.1. A birth-death process with rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$ is exponentially ergodic if and only if $\sigma(\mathcal{P}) > 0$.

PROOF. From (2.23), (2.42) and Theorem 3.3 one has

$$(5.1) \qquad \sigma(\mathcal{P}) \geqq \max\{\xi_1(\mathcal{P}), \xi_1(\mathcal{P}^d)\} \geqq \gamma(d\psi) \ ,$$

so that $\sigma(\mathcal{P}) > 0$ if $\gamma(d\psi) > 0$. On the other hand, if $\gamma(d\psi) = 0$, then 0 must be a limit point of $S(d\psi)$, whence $\sigma(\mathcal{P}) = 0$ by Theorem 2.2. □

Defining

$$(5.2) \qquad \mathcal{P}^{(k)} = \{\lambda_{n+k}, \mu_{n+k}\}_{n=0}^{\infty}$$

$(k = 1,2,\ldots)$, we obtain from Chihara (1978), Theorem III.4.2

$$(5.3) \qquad \sigma(\mathcal{P}^{(k)}) = \sigma(\mathcal{P})$$

(k = 1,2,...), so that the next corollary, noted earlier by Callaert and Keilson (1973a), emerges.

COROLLARY 5.2. Any finite number of changes in the rate values of a birth-death process does not affect the prevalence (or non-prevalence) of exponential ergodicity.

Of course, (5.1) and Theorem 5.1 imply

$$(5.4) \qquad \sigma(\mathcal{P}) > 0 \iff \xi_1(\mathcal{P}) > 0 \quad \text{or} \quad \xi_1(\mathcal{P}^d) > 0.$$

Thus from considerations of symmetry we have

$$(5.5) \qquad \sigma(\mathcal{P}) > 0 \iff \sigma(\mathcal{P}^d) > 0,$$

so that a birth-death process is exponentially ergodic if and only if its dual process is exponentially ergodic.

We remark also that by Chihara (1978), Theorem III.4.2

$$(5.6) \qquad \xi_1(\mathcal{P}^{(1)}) \geqq \xi_1(\mathcal{P}).$$

Further, it may be shown from Proposition B on p.394 of Karlin and McGregor (1957b) (and under certain restrictions, which do not affect the general validity of (5.8), from Chihara (1957), Theorem 3) that

$$(5.7) \qquad \sigma(\mathcal{P}) > \xi_1(\mathcal{P}) \implies \xi_1(\mathcal{P}^{(1)}) > \xi_1(\mathcal{P}) .$$

It follows that

$$(5.8) \qquad \sigma(\mathcal{P}) > 0 \iff \xi_1(\mathcal{P}^{(1)}) > 0,$$

which leads to yet another necessary and sufficient condition for exponential ergodicity. All these conditions as such, however, add little to Theorem 5.1, or indeed, to Theorem 3.3, since the representations that exist for the quantities concerned ($\sigma(\mathcal{P})$, $\xi_1(\mathcal{P})$, $\xi_1(\mathcal{P}^d)$, etc.) are structurally similar (cf. Chihara (1978), Van Doorn (1982)). It is interesting to note that characterizations of the type (3.8) lead to necessary and sufficient conditions for exponential ergodicity which are similar to Tweedie's (1981) criterion. His result, specified for birth-death processes, is one of the many forms in which a criterion may be moulded.

5.2 Birth-death processes with rational rates

In this section we will show that $\sigma(\mathcal{P})$ can be expressed in the elements of $\mathcal{P} = \{\lambda_n, \mu_n\}$ when these rates are rational functions of n from some finite value of n onwards. We assume

$$(5.9) \qquad \lambda_n = \frac{a_0 n^p + a_1 n^{p-1} + O(n^{p-2})}{b_0 n^q + b_1 n^{q-1} + O(n^{q-2})}, \quad \mu_n = \frac{c_0 n^r + c_1 n^{r-1} + O(n^{r-2})}{d_0 n^s + d_1 n^{s-1} + O(n^{s-2})}$$

as $n \to \infty$, and suppose that a_0, b_0, c_0, $d_0 > 0$, which is no essential restriction. (Of course, λ_n and μ_n must be positive except $\mu_0 \geqq 0$).

THEOREM 5.3. If a rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$ satisfies (5.9) as $n \to \infty$, then the following hold:

(i) $p \leqq q$ and $r \leqq s \Rightarrow \sigma(\mathcal{P}) = \left(\left(\frac{a_0}{b_0}\right)^{\frac{1}{2}} - \left(\frac{c_0}{d_0}\right)^{\frac{1}{2}}\right)^2$,

(ii) $(p > q$ or $r > s)$ and $(p - q = r - s \leqq 2 \Rightarrow a_0 d_0 \neq b_0 c_0)$
$\Rightarrow \sigma(\mathcal{P}) = \infty$,

(iii) $p - q = r - s = 1$ and $a_0 d_0 = b_0 c_0 \Rightarrow \sigma(\mathcal{P}) = 0$,

(iv) $p - q = r - s = 2$ and $a_0 d_0 = b_0 c_0 \Rightarrow$

$$\sigma(\mathcal{P}) = \tfrac{1}{4}\,\frac{a_0}{b_0}\left(\frac{a_1}{a_0} - \frac{b_1}{b_0} - \frac{c_1}{c_0} + \frac{d_1}{d_0} - 1\right)^2 .$$

PROOF. Blumenthal's Theorem (see Chihara (1978), p. 121) yields (i).

Statement (ii) follows from Maki (1976), Theorem 8, except for the case $p-q=r-s>2$ and $a_0d_0=b_0c_0$.

Using (3.5) and (3.6) and after straightforward but tedious calculations, the remaining cases can be reduced to situations solved by Chihara (1982b), Theorems 5.1 and 5.2. □

In view of Theorem 5.1, the above result enables us to decide in a very simple way on the prevalence of exponential ergodicity for the vast majority of birth-death processes encountered in practice.

5.3 Sufficient conditions

We see from Theorem 3.3 (ii) that a sufficient condition for exponential ergodicity of a birth-death process associated with a rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$ is given by

$$\sum (\pi_n + (\lambda_n \pi_n)^{-1}) < \infty .$$

This condition can be sharpened somewhat. For we clearly have $\sigma(\mathcal{P}) = \infty$ if $\xi_2(\mathcal{P}) > 0$ and $\sum_{i>1} \xi_i^{-1}(\mathcal{P}) < \infty$. The necessary and sufficient condition for the latter to hold is given in Theorem A.3 of Appendix 1.

<u>THEOREM 5.4.</u> A birth-death process with rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$ is exponentially ergodic if at least one of the series

$$\sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} \sum_{i=0}^{n} \pi_i \tag{5.10}$$

and

$$\sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} \sum_{i=n+1}^{\infty} \pi_i \tag{5.11}$$

is convergent.

This result was given earlier for the case $\mu_0 = 0$ by Callaert and Keilson (1973b) and (partly) by Tweedie (1981).

REMARK. The series (5.10) and (5.11) arise frequently in studies of birth-death processes, particularly those where either the forward or the backward equations are omitted from the postulates (2.1) - (2.7) (see, e.g., Reuter (1957), Feller (1959), Kemperman (1962); for probabilistic interpretations see John (1957), Keilson (1965) and Callaert and Keilson (1973a)).

An entirely different approach to obtaining simple sufficient conditions for exponential ergodicity of a birth-death process with rate set $\mathcal{P}$ is looking at lower bounds for $\sigma(\mathcal{P})$. It is clear from (5.3) that these can be obtained from the lower bounds for $\xi_1(\mathcal{P})$ given in Section 4.1 by inserting $\lim_{n\to\infty}$ before inf. Consider for instance the bound (4.1). Its counterpart for $\sigma(\mathcal{P})$ reads

$$(5.12) \qquad \sigma(\mathcal{P}) \geqq \lim_{n\to\infty} \inf \{\lambda_n + \mu_n - (\lambda_{n-1}\mu_n)^{\frac{1}{2}} - (\lambda_n\mu_{n+1})^{\frac{1}{2}}\} .$$

Combining this bound and Theorem 5.1 yields

<u>THEOREM 5.5.</u> A birth-death process with rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$ is exponentially ergodic if

$$(5.13) \qquad \lim_{n\to\infty} \inf \{\lambda_n + \mu_n - (\lambda_{n-1}\mu_n)^{\frac{1}{2}} - (\lambda_n\mu_{n+1})^{\frac{1}{2}}\} > 0 .$$

Similar theorems may be formulated on the basis of the other lower bounds for $\xi_1(\mathcal{P})$ given in Section 4.1, but we will not pursue this here.

Our third approach is to look at characterizations for $\sigma(\mathcal{P})$, in particular the analogue of (3.10), which reads

(5.14) $$\sigma(\mathcal{P}) = \sup\{x \mid x < \lambda_n + \mu_n \text{ for all } n, \text{and } \{\alpha_n(x)\}_{n=N}^{\infty} \text{ is a chain sequence for } N \text{ sufficiently large}\}$$

(Chihara (1978), Theorem IV.3.2), where $\alpha_n(x)$ is given by (3.9). Thus for $\sigma(\mathcal{P})$ to be positive it is necessary and sufficient that there exists an $\varepsilon > 0$ such that

(5.15) $$\varepsilon < \lambda_n + \mu_n$$

$(n = 0,1,\ldots)$ and

(5.16) $$\{\alpha_n(\varepsilon)\}_{n=N}^{\infty} \text{ is a chain sequence for } N \text{ sufficiently large.}$$

Considering that $\{\frac{1}{4}\}$ is a chain sequence, it follows by Wall's Comparison Theorem (Chihara (1978), Theorem III.5.7) that for (5.16) to hold it is sufficient that

(5.17) $$\limsup_{n\to\infty} \{\alpha_n(\varepsilon)\} \leq \tfrac{1}{4}.$$

A sufficient condition for (5.17) to be valid for some $\varepsilon > 0$ is clearly given by

(5.18) $$\limsup_{n\to\infty} \{\alpha_n(0)\} < \tfrac{1}{4},$$

so that we obtain the next theorem, which was stated earlier by Callaert and Keilson for the case $\mu_0 = 0$.

<u>THEOREM 5.6.</u> A birth-death process with rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$ is exponentially ergodic if

(5.19) $$\liminf_{n\to\infty} \{\lambda_n + \mu_n\} > 0 \text{ and } \limsup_{n\to\infty} \{\frac{\lambda_{n-1}\mu_n}{(\lambda_{n-1}+\mu_{n-1})(\lambda_n+\mu_n)}\} < \tfrac{1}{4} .$$

REMARK. Chihara (1982b) gives detailed information on the value of $\sigma(\mathcal{P})$ when $\lambda_n + \mu_n \to \infty$ and $\alpha_n(0) \to \frac{1}{4}$.

Most exponentially ergodic birth-death processes in practical models will satisfy the criterion given in the next corollary, given earlier by Callaert and Keilson (1973b) for $\mu_0 = 0$.

<u>COROLLARY 5.7.</u> A birth-death process with rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$ is exponentially ergodic if

(5.20) $$\liminf_{n\to\infty} \{\lambda_n + \mu_n\} > 0 \quad \text{and} \quad \lim_{n\to\infty} \lambda_n/\mu_n = \theta \neq 1 .$$

<u>5.4 Necessary conditions</u>

From Lemma 3.2 (iv) we immediately obtain a simple necessary condition for exponential ergodicity of a birth-death process.

<u>THEOREM 5.8.</u> A birth-death process with rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$ is exponentially ergodic only if

(5.21) $$\sum \pi_n < \infty \quad \text{or} \quad \sum (\lambda_n \pi_n)^{-1} < \infty .$$

Let us now look at upper bounds for $\sigma(\mathcal{P})$. These may be obtained from the upper bounds for $\xi_1(\mathcal{P})$ given in Section 4.2 by inserting $\liminf_{n\to\infty}$ before the bound (and writing $\leqq$ instead of $<$), as is evident from (5.3). For instance, (4.7) gives the bound

$$\sigma(\mathcal{P}) \leqq \liminf_{n\to\infty} \{\lambda_n + \mu_n\} \ , \tag{5.22}$$

whence it is necessary that

$$\liminf_{n\to\infty} \{\lambda_n + \mu_n\} > 0 \tag{5.23}$$

for a birth-death process with rate set $\mathcal{P} = \{\lambda_n,\mu_n\}$ to be exponentially ergodic (this result is also given in Callaert and Keilson (1973a)). Similar results may be obtained on the basis of other bounds for $\xi_1(\mathcal{P})$. Again, we will not spell them out. However, we give one result that is based on an upper bound for $\sigma(\mathcal{P})$ which has no direct counterpart for $\xi_1(\mathcal{P})$. Namely, from Van Doorn (1982), Corollary 6.1 we have

$$\sigma(\mathcal{P}) \leqq \liminf_{k\to\infty} \{\frac{1}{k} \sum_{n=0}^{k} (\lambda_n + \mu_n - 2(\lambda_{n-1}\mu_n)^{\frac{1}{2}})\} \tag{5.24}$$

$(\lambda_{-1} \equiv 0)$. Hence the following holds.

<u>THEOREM 5.9.</u> A birth-death process with rate set $\mathcal{P} = \{\lambda_n,\mu_n\}$ is exponentially ergodic only if

$$\liminf_{k\to\infty} \{\frac{1}{k} \sum_{n=0}^{k} (\lambda_n + \mu_n - 2(\lambda_{n-1}\mu_n)^{\frac{1}{2}})\} > 0 \ . \tag{5.25}$$

6. Examples

To assess the relative merit of the bounds given in Chapter 4 we shall evaluate them for some birth-death processes for which the decay parameter is known exactly. The particular examples have been chosen because of the markedly different types of their spectra. In each case the rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$ is such that the uniqueness condition $\sum(\pi_n + (\lambda_n \pi_n)^{-1}) = \infty$ holds, as can easily be verified. Also, in each case we have $\mu_0 = 0$, so that the decay parameter equals $\xi_1(\mathcal{P}^d)$. Thus we compute the bounds in Chapter 4 for the dual rates.

1. Our first example concerns the queue length process of the M/M/s queue, which is a birth-death process with rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$ satisfying

$$(6.1) \qquad \lambda_n = \lambda \ , \quad \mu_n = \mu \min(n,s)$$

$(n = 0,1,\ldots)$, where λ, $\mu > 0$ and s is a positive integer. From Theorem 5.3 (i) we have

$$(6.2) \qquad \sigma(\mathcal{P}) = (\lambda^{\frac{1}{2}} - (s\mu)^{\frac{1}{2}})^2 ,$$

so that the process is exponentially ergodic if and only if $\lambda \neq s\mu$, as is well known. The decay parameter $\gamma(\mathcal{P})$ of this process is calculated as follows (cf. Van Doorn (1981a), Theorem 6.2.13). Let $\rho = \lambda/s\mu$ denote the traffic intensity and define

(6.3) $$C(x) = \tfrac{1}{2}[1 - x + \frac{1}{\rho} - ((1-x+\frac{1}{\rho})^2 - \frac{4}{\rho})^{\frac{1}{2}}]$$

and

(6.4) $$R_1(x,y) = 1 - x$$
$$R_{n+1}(x,y) = 1 - x + \frac{n}{sy} - \frac{n}{syR_n(x,y)}, \quad n = 1,2,\ldots,s-1.$$

Finally, let ρ^* be the largest root < 1 of the equation

(6.5) $$R_s(1-y^{-\frac{1}{2}},y) = y^{-\frac{1}{2}}$$

if $s > 1$, and $\rho^* = 0$ if $s = 1$. Then, if $\rho \geqq \rho^*$

(6.6) $$\gamma(\mathcal{P}) = (\lambda^{\frac{1}{2}} - (s\mu)^{\frac{1}{2}})^2,$$

whereas for $\rho < \rho^*$, $\gamma(\mathcal{P})$ equals λ times the smallest positive root of the equation

(6.7) $$R_s(x,\rho) = C(x).$$

We let $\mu = 1$ and $s = 10$. The critical value ρ^* for the traffic intensity then equals .498. The second column in Table 6.1 contains the exact values (up to three decimal places) of the decay parameter for several values of the traffic intensity. The third, fourth and fifth column contain lower bounds for $\gamma(\mathcal{P})$ obtained by Corollaries 4.2, 4.3 and 4.5, respectively. The sixth and seventh column contain upper bounds for $\gamma(\mathcal{P})$ obtained from (4.6) by taking $\chi_n = 1$ and $\chi_n = \nu_n$, respectively. Finally, the eighth column gives the results of

Corollary 4.7. The best lower and upper bounds have been underlined.

2. The second example is a birth-death process which models the queue length process of a queueing system where potential customers are discouraged by queue length. Its rate set $\mathcal{P}$ = $\{\lambda_n, \mu_n\}$ satisfies $\mu_0 = 0$ and

(6.8) $$\lambda_n = \lambda/(n+1) \ , \quad \mu_{n+1} = \mu$$

(n = 0,1,...), where λ, $\mu > 0$. Theorem 5.3 (i) yields

(6.9) $$\sigma(\mathcal{P}) = \mu,$$

so that the process is exponentially ergodic. Indeed, the decay parameter for this process is given by

(6.10) $$\gamma(\mathcal{P}) = \mu - \tfrac{1}{2}[(\lambda^2+4\lambda\mu)^{\frac{1}{2}} - \lambda] > 0$$

(Van Doorn (1981b)). Table 6.2 shows the results that were obtained by calculations similar to those of the previous example. Again we have taken $\mu = 1$.

3. The final example is a linear growth birth-death process with rate set $\mathcal{P} = \{\lambda_n, \mu_n\}$, where

(6.11) $$\lambda_n = \lambda(n+1) \ , \quad \mu_n = \mu n$$

$(n = 0,1,\ldots)$, with λ, $\mu > 0$. This model is of some interest in a queueing context (cf. Conolly (1975)). Theorem 5.3 (ii) and (iii) imply

$$(6.12) \qquad \sigma(\mathcal{P}) = \begin{cases} 0 & \text{if } \lambda = \mu \\ \infty & \text{if } \lambda \neq \mu. \end{cases}$$

Thus the process is exponentially ergodic if and only if $\lambda \neq \mu$. Indeed, according to Karlin and McGregor (1958), the decay parameter of this process is given by

$$(6.13) \qquad \gamma(\mathcal{P}) = |\lambda - \mu|.$$

Table 6.3 shows the results we obtained for this example with $\mu = 1$.

We conclude with some general remarks concerning the examples. Most calculations can be done by hand (and a simple pocket calculator) by straightforward but sometimes cumbersome manipulations. As is to be expected, the results depend strongly on the specific form of the birth and death rates. But, unfortunately, no clear rules can be elicited from the examples prescribing which bounds to use for certain types of rates. It is clear though from the limited amount of numerical information we have gathered, that all bounds considered are of potential significance.

		lower bounds			upper bounds		
$\rho=\lambda/10$	γ	Cor. 4.2	Cor. 4.3	Cor. 4.5	$(4.6)_{\chi\equiv 1}$	$(4.6)_{\chi\equiv\nu}$	Cor. 4.7
.05	1.000	.793	.944	.500	1.293	1.178	1.134
.1	1.000	.586	.889	.438	1.391	1.241	1.382
.2	1.000	.551	.778	.469	1.523	1.377	2.000
.3	.998	.536	.667	.479	1.640	1.515	2.697
.4	.984	.528	.556	.484	1.351	1.351	3.438
.5	.858	.523	.444	.488	.858	.858	4.209
.6	.508	.508	.333	.490	.508	.508	5.000
.7	.267	.267	.222	.267	.267	.267	5.725
.8	.111	.111	.111	.111	.111	.111	6.469
.9	.026	.026	.026	.026	.026	.026	7.228
1.0	.000	<0	.000	.000	.000	-.000	8.000
1.1	.024	<0	.024	.024	.024	.024	8.734
1.2	.091	<0	.091	.091	.091	.091	9.479
1.3	.196	<0	.196	.196	.196	.196	10.235
1.4	.336	<0	.336	.336	.336	.336	11.000
1.5	.505	.134	.505	.505	.505	.505	11.738

Table 6.1. The exact value and bounds for the decay parameter γ of the queue length process of the M/M/10 queue with service intensity 1 and arrival intensity λ.

		lower bounds			upper bounds		
λ	γ	Cor. 4.2	Cor. 4.3	Cor. 4.5	$(4.6)_{\chi\equiv 1}$	$(4.6)_{\chi\equiv\nu}$	Cor. 4.7
.2	.642	.437	.468	.516	.688	.686	.825
.4	.537	.320	.306	.400	.603	.596	.800
.6	.469	.278	.205	.344	.551	.540	.787
.8	.420	.251	.135	.298	.513	.498	.780
1.0	.382	.216	.086	.259	.478	.466	.775
1.2	.351	.193	.051	.231	.450	.440	.771
1.4	.325	.180	.027	.212	.427	.419	.769
1.6	.303	.171	.011	.200	.408	.398	.767
1.8	.284	.155	.003	.181	.392	.380	.765
2.0	.268	.143	.000	.167	.378	.364	.764
2.2	.253	.135	.000	.156	.364	.351	.763
2.4	.240	.131	.000	.148	.351	.338	.762
2.6	.229	.123	.000	.141	.339	.327	.761
2.8	.218	.115	.000	.132	.329	.317	.760
3.0	.209	.109	.000	.124	.319	.308	.760

Table 6.2. The exact value and bounds for the decay parameter γ of the birth-death process with rates $\mu_0 = 0$ and $\mu_{n+1} = 1$, $\lambda_n = \lambda/(n+1)$ $(n \geq 0)$.

λ	γ	lower bounds			upper bounds		
		Cor. 4.2	Cor. 4.3	Cor. 4.5	$(4.6)_{\chi\equiv 1}$	$(4.6)_{\chi\equiv\nu}$	Cor. 4.7
.05	.950	.734	$-\infty$	.753	1.050	1.050	.962
.1	.900	.653	$-\infty$	.600	1.100	1.045	.941
.2	.800	.568	$-\infty$	.400	1.168	.988	.928
.3	.700	.484	$-\infty$	.270	1.175	.919	.939
.4	.600	.356	$-\infty$	.179	1.171	.847	.964
.5	.500	.268	$-\infty$	.114	1.159	.773	1.000
.6	.400	.207	$-\infty$	.068	1.139	.695	1.044
.7	.300	.152	$-\infty$	.036	1.114	.612	1.093
.8	.200	.101	$-\infty$	.015	1.083	.521	1.148
.9	.100	.050	$-\infty$	.004	1.057	.410	1.206
1.0	.000	.000	$-\infty$	.000	1.000	.000	1.268
1.1	.100	.050	$-\infty$	.003	1.163	.438	1.333
1.2	.200	.101	$-\infty$	.012	1.287	.585	1.400
1.3	.300	.151	$-\infty$	.026	1.421	.716	1.469
1.4	.400	.203	$-\infty$	.045	1.554	.840	1.541
1.5	.500	.257	$-\infty$	.067	1.684	.961	1.641

Table 6.3. The exact value and bounds for the decay parameter γ of the birth-death process with rates $\lambda_n = \lambda(n+1)$ and $\mu_n = n$.

Appendices

In these appendices we shall refer to the work of Karlin and McGregor (1957a) by the letters KM followed by the pertinent page, theorem or formula.

1. Convergence of birth-death polynomials

Here we state some results which are used in Chapter 5 and Appendix 4 (a little more, actually). We will adopt the convention that $\sum_i' a_i^{-1}$ denotes the series obtained after omitting from $\{a_i\}$ any terms that are equal to zero; if $a_i = 0$ for all i we let $\sum_i' a_i^{-1} = \infty$.

LEMMA A.1. For any set $\mathcal{P} = \{\lambda_n, \mu_n\}$ of birth and death rates the following statements are equivalent:

(A.1) The sequence of polynomials $\{Q_n(x)\}$ associated with $\mathcal{P}$ converges on bounded sets to an entire function whose zeros are precisely the points $\xi_i(\mathcal{P})$, $i = 1,2,\ldots,$ where $\lim_{i\to\infty} \xi_i(\mathcal{P}) \equiv \sigma(\mathcal{P}) = \infty$.

(A.2) $$\sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} \sum_{i=0}^{n} \pi_i < \infty .$$

LEMMA A.2. For any set $\mathcal{P} = \{\lambda_n, \mu_n\}$ of birth and death rates the following statements are equivalent, where a is any number smaller than $\xi_1(\mathcal{P})$:

(A.3) The sequence $\{Q_n(x)/Q_n(a)\}$ converges on bounded sets to an entire function whose zeros are precisely the points $\xi_i(\mathcal{P})$, $i = 1,2,\ldots,$ where $\lim_{i\to\infty} \xi_i(\mathcal{P}) \equiv \sigma(\mathcal{P}) = \infty$.

(A.4) $$\sum_i' \xi_i^{-1}(\mathcal{P}) < \infty .$$

Lemma A.1 (partly given in KM, Lemma 4) is due to Stieltjes (1918), pp. 524-527. This can be seen by expressing Stieltjes' parameters a_n in the rates λ_n and μ_n as follows

$$\text{(A.5)} \qquad \mu_0 = 0 \implies a_{2n+1} = \pi_n, \; a_{2n+2} = (\lambda_n \pi_n)^{-1},$$

the polynomial $Q_n(x)$ associated with $\mathcal{P}$ is then identical to Stieltjes' polynomial $Q_{2n}(-x)$ ($n = 0,1,\ldots$), and

$$\text{(A.6)} \qquad \mu_0 > 0 \implies a_{2n+2} = \pi_n/\mu_0, \; a_{2n+3} = \mu_0(\lambda_n \pi_n)^{-1}$$

($a_1 = 1$), the polynomial $Q_n(x)$ can then be identified with Stieltjes' polynomial $Q_{2n}(-x)/(-x)$ ($n = 0,1,\ldots$).

Lemma A.2 was proven by Chihara (1972).

We can now give a precise criterion in terms of $\mathcal{P}$ for convergence of the series $\sum_i' \xi_i^{-1}(\mathcal{P})$.

<u>THEOREM A.3</u>. For any set $\mathcal{P} = \{\lambda_n, \mu_n\}$ of birth and death rates one has $\sum_i' \xi_i^{-1}(\mathcal{P}) < \infty$ if and only if

$$\text{(A.7)} \qquad \sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} \sum_{i=0}^{n} \pi_i < \infty \quad \text{or} \quad \sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} \sum_{i=n+1}^{\infty} \pi_i < \infty .$$

PROOF. Suppose that (A.2) (hence (A.1)) holds. We then have $\sum(\lambda_n \pi_n)^{-1} < \infty$ and $\sigma(\mathcal{P}) = \infty$, whence $\xi_1(\mathcal{P}) > 0$, by Lemma 3.2 (ii) and (iii). From the recurrence relations (2.17) we easily obtain

$$\text{(A.8)} \qquad Q_n(0) = 1 + \mu_0 \sum_{i=0}^{n-1} (\lambda_i \pi_i)^{-1} .$$

Therefore, if $\mu_0 = 0$ then (A.1) implies (A.3) for a = 0, so that (A.4) holds. If, on the other hand, $\mu_0 > 0$, then $Q_n(0)$ is increasing with n but bounded, so that (A.3) is valid again for a = 0. Summarizing

(A.9) $$\sum_{n=0}^{\infty}(\lambda_n\pi_n)^{-1}\sum_{i=0}^{n}\pi_i < \infty \quad \Rightarrow \quad \sum_i{}' \xi_i^{-1}(\mathcal{P}) < \infty .$$

In view of (2.42) we have

(A.10) $$\sum_i{}' \xi_i^{-1}(\mathcal{P}) < \infty \quad \Leftrightarrow \quad \sum_i{}' \xi_i^{-1}(\mathcal{P}^d) < \infty .$$

Therefore, a sufficient condition for (A.4) is also given by (A.2) formulated in terms of the dual parameters. Considering that

(A.11) $$\sum_{n=0}^{\infty}(\lambda_n^d\pi_n^d)^{-1}\sum_{i=0}^{n}\pi_i^d < \infty \quad \Leftrightarrow \quad \sum_{n=0}^{\infty}\pi_{n+1}\sum_{i=0}^{n}(\lambda_i\pi_i)^{-1} < \infty$$

by (2.43), and

(A.12) $$\sum_{n=0}^{\infty}\pi_{n+1}\sum_{i=0}^{n}(\lambda_i\pi_i)^{-1} < \infty \quad \Leftrightarrow \quad \sum_{n=0}^{\infty}(\lambda_n\pi_n)^{-1}\sum_{i=n+1}^{\infty}\pi_i < \infty,$$

the sufficiency has now been established.

To prove the necessity we argue as follows. Suppose (A.4) is valid. Then, apparently $\xi_i(\mathcal{P}) \to \infty = \sigma(\mathcal{P})$ as $i \to \infty$, so that, by Lemma 3.2 (iv) either $\sum\pi_n < \infty$ or $\sum(\lambda_n\pi_n)^{-1} < \infty$.

First suppose $\mu_0 > 0$. We then have $\xi_1(\mathcal{P}) > 0$, by Lemma 3.2 (iii), so that (A.3) is valid with a = 0. If $\sum(\lambda_n\pi_n)^{-1} < \infty$, then $Q_n(0)$ is bounded, so that (A.1) holds. It follows that the left inequality of (A.7) prevails. If, on the other hand, $\sum(\lambda_n\pi_n)^{-1} = \infty$, we must have $\sum\pi_n < \infty$, or equivalently, $\sum(\lambda_n^d\pi_n^d)^{-1} < \infty$. Hence, considering that $\mu_0^d = 0$ and using (5.5) and Lemma 3.2 (ii), we have $\xi_1(\mathcal{P}^d) > 0$, so that (A.3) holds with $a = 0$ in terms of the dual parameters. Since $Q_n^d(0) = 1$, (A.1) then holds as well for the dual parameters, implying (A.2) in dual terms, which, by (A.11) and (A.12), amounts to the right inequality of (A.7).

If $\mu_0 = 0$ we can repeat the above argument in dual terms in view of (A.10). This completes the proof. □

2. Proof of Theorem 2.1

Consider a pair $\{\mu_0, d\psi\}$ satisfying the conditions (2.11)-(2.16) and let

$$p_{ij}(t,d\psi) = \pi_j \int_0^\infty e^{-xt} Q_i(x)Q_j(x)d\psi(x) \tag{A.13}$$

$(i,j \in E,\ t \geqq 0)$, where π_n and Q_n are defined as in Theorem 2.1. By KM, Theorems 2, 4 and 9 the set $\{p_{ij}(t,d\psi)\}$ satisfies the conditions (2.2)-(2.7); by KM, Theorem 5 it satisfies (2.10) if $\mu_0 = 0$, while by KM, Theorem 7 the set satisfies

$$\sum_j p_{ij}(t,d\psi) \leqq 1 \tag{A.14}$$

$(i \in E,\ t \geqq 0)$, if $\mu_0 > 0$. The latter statement can be strengthened somewhat for by (A.13) and Fubini's theorem we have

$$\text{(A.15)}\qquad \mu_0 \int_0^t p_{i0}(\tau,d\psi)d\tau = \mu_0 \int_0^t\{\int_0^t e^{-xt}Q_i(x)d\psi(x)\}d\tau$$

$$= \mu_0 \int_0^\infty Q_i(x)d\psi(x)/x - \mu_0 \int_0^\infty e^{-xt}Q_i(x)d\psi(x)/x \ ,$$

where both terms on the right hand side are finite by conditions (2.15) and (2.16). Adding (A.15) to both sides of KM, (3.12) and subsequently using part of the argument on KM, p. 513 readily yields that the set $\{p_{ij}(t,d\psi)\}$ satisfies (2.10).

Now consider a set $\{p_{ij}(t) \mid i,j \in E,\ t \geqq 0\}$ satisfying (2.2)-(2.7) and (2.10). Then, by KM, Theorem 12, there exists a unique pair $\{\mu_0,d\psi\}$ satisfying (2.11)-(2.15), such that $p_{ij}(t) = p_{ij}(t,d\psi)$ $(i,j \in E,\ t \geqq 0)$, where $p_{ij}(t,d\psi)$ is given by (A.13). To show that $\{\mu_0,d\psi\}$ satisfies (2.16), we discern three cases. First, if $d\psi$ is uniquely determined by its moments, then the validity of (2.16) is immediately implied by KM, Lemma 2. Therefore suppose that $d\psi$ is not uniquely determined by its moments. Then, since $d\psi$ is extremal, its spectrum is discrete and the first point ξ in $S(d\psi)$ is a point in the interval $[0,\xi_1]$, where ξ_1 is some positive number (KM, p. 501; see also the proof of Theorem 2.2). Supposing $\xi > 0$, we have by (2.10) and Fubini's theorem

$$1 \geqq \mu_0 \int_0^t p_{00}(\tau)d\tau = \mu_0 \int_0^t p_{00}(\tau,d\psi)d\tau =$$

$$\mu_0 \int_0^\infty(1-e^{-xt})d\psi(x)/x = \mu_0 \int_\xi^\infty d\psi(x)/x - \mu_0 \int_\xi^\infty e^{-xt}d\psi(x)/x,$$

from which (2.16) readily follows by letting t tend to infinity.

Finally, supposing $\xi = 0$, the argument on KM, p. 514 shows that

$$\sum_{j=0}^{n} p_{ij}(t) > 1$$

for n, i and t sufficiently large, which is a contradiction. □

3. $\phi(\xi)$ is decreasing

We are dealing with the situation where the Stieltjes moment problem associated with a set $\mathcal{P} = \{\lambda_n, \mu_n\}$ of birth and death rates, where $\mu_0 > 0$, has no unique solution. The extremal solutions of the moment problem are $d\psi_\xi$, $0 \leq \xi \leq \xi_1(\mathcal{P})$. Now let $0 < \xi < \xi_1$ and $\phi(\xi)$ as in (2.33). Then, by KM, p. 530,

$$\text{(A.16)} \qquad \phi(\xi) = -\mu_0 \frac{H^{(0)}(0)Q(\xi) - H(\xi)Q^{(0)}(0)}{H(0)Q(\xi) - H(\xi)Q(0)} ,$$

where H, $H^{(0)}$, Q and $Q^{(0)}$ are entire functions and $H(0)Q(\xi) \neq H(\xi)Q(0)$ (these functions are of the type of the limit function in Lemma A.1). Hence,

$$\text{(A.17)} \qquad \phi'(\xi) = -\mu_0\{H(\xi)Q'(\xi)-H'(\xi)Q(\xi)\} \times \{H(0)Q^{(0)}(0)-H^{(0)}(0)Q(0)\}/D^2 ,$$

where D denotes the denominator in (A.16). By KM, (2.38)

$$\text{(A.18)} \qquad H(0)Q^{(0)}(0) - H^{(0)}(0)Q(0) = 1,$$

while by KM, (2.37)

(A.19)
$$-\sum_{n=0}^{\infty} Q_n(x)Q_n(y)\pi_n + (x-y)\sum_{n=0}^{\infty} Q_n(x)Q_n'(y)\pi_n$$
$$= H'(y)Q(x) - H(x)Q'(y) ,$$

so that

(A.20)
$$H(\xi)Q'(\xi) - H'(\xi)Q(\xi) = \sum_{n=0}^{\infty} \pi_n Q_n^2(\xi) .$$

Thus

(A.21)
$$\phi'(\xi) = - \mu_0 \sum_{n=0}^{\infty} \pi_n Q_n^2(\xi)/D^2 < 0 .$$

Finally, $\phi(\xi_1(\mathcal{P})) \leqq 1$ by KM, (2.4) and KM, Lemma 6, while $\phi(\xi) \to \infty$ as $\xi \to 0$, since the mass of $d\psi_\xi$ at ξ, $0 \leqq \xi \leqq \xi_1(\mathcal{P})$ is bounded away from zero by Lemma 2.3.

4. Proof of Theorem 2.4 (ii)

We state some preliminary results first. If for a set $\mathcal{P} = \{\lambda_n, \mu_n\}$ of birth and death rates the series (2.29) converges, then, evidently, the series

(A.22)
$$\sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} \sum_{i=0}^{n} \pi_i$$

and

(A.23)
$$\sum_{n=C}^{\infty} (\lambda_n \pi_n)^{-1} \sum_{i=n+1}^{\infty} \pi_i$$

both converge. Hence, in view of (A.11) and (A.12), condition

(A.2) of Lemma A.1 applies to both $\mathcal{P}$ and the dual set of rates $\mathcal{P}^d = \{\lambda_n^d, \mu_n^d\}$, implying that

$$Q_n(x) \to Q_\infty(x) \quad \text{and} \quad Q_n^d(x) \to Q_\infty^d(x)$$

as $n \to \infty$, where $Q_\infty(x)$ and $Q_\infty^d(x)$ are entire functions whose zeros are precisely the points $\xi_i(\mathcal{P})$ and $\xi_i(\mathcal{P}^d)$, respectively. By (2.42) the zeros $\xi_i(\mathcal{P})$ and $\xi_i(\mathcal{P}^d)$ interlace. In fact, strict inequalities hold as shown by KM, p. 506 (in their notation either $Q_n^d(x) = H_{n+1}(x)/(-x)$ or $Q_n^d(x) = \mu_0 H_n(x)$). The latter result can also be seen from Chihara (1982a) after making some suitable substitutions.

From (2.17), (2.41) and (2.43) it can straightforwardly be verified that for $n \geqq 0$

$$\sum_{k=0}^{n} \pi_k Q_k^2(\xi) = \frac{\xi}{\lambda_0} \sum_{k=0}^{n} \pi_k^d (Q_k^d(\xi))^2 + Q_{n+1}(\xi) Q_n^d(\xi)$$

if $\mu_0 = 0$, and

$$\frac{\xi}{\mu_0} \sum_{k=0}^{n} \pi_k Q_k^2(\xi) = \sum_{k=0}^{n} \pi_k^d (Q_k^d(\xi))^2 - Q_n(\xi) Q_{n+1}^d(\xi)$$

if $\mu_0 > 0$. Letting n tend to infinity yields

(A.24) $$\mu_0 = 0 \Rightarrow \sum_{k=0}^{\infty} \pi_k Q_k^2(\xi) = \frac{\xi}{\lambda_0} \sum_{k=0}^{\infty} \pi_k^d (Q_k^d(\xi))^2 + Q_\infty(\xi) Q_\infty^d(\xi)$$

(A.25) $$\mu_0 > 0 \Rightarrow \frac{\xi}{\mu_0} \sum_{k=0}^{\infty} \pi_k Q_k^2(\xi) = \sum_{k=0}^{\infty} \pi_k^d (Q_k^d(\xi))^2 - Q_\infty(\xi) Q_\infty^d(\xi) ,$$

and each term is finite by Lemma 2.3.

Turning finally to the proof of Theorem 2.4 (ii), we note that by KM, Lemmas 2 and 3 the right hand sides of (2.46) and (2.47) determine distributions corresponding to $\mathcal{P}$ and $\mathcal{P}^d$ which satisfy the conditions (2.12) - (2.16) except, perhaps, for the extremality condition (2.13). Thus our task is to show that the right hand sides of (2.46) and (2.47) are extremal and have their first spectral points on the edge of the allowed interval. We discern four cases.

i. Let $\mu_0 = 0$ and

$$d\chi(x) = x d\psi_{max}(x)/\lambda_0 \ .$$

Also, let ξ^* denote the first positive spectral point of $d\psi_{max}$ (we know that 0 is the first spectral point of $d\psi_{max}$ and $0 < \xi_1(\mathcal{P}) < \xi^* < \xi_2(\mathcal{P})$). Since $d\psi_{max}$ is extremal, the mass of $d\psi_{max}$ at the point ξ^* equals $\{\sum \pi_k Q_k^2(\xi^*)\}^{-1}$, according to Lemma 2.3, so that by (A.24)

$$d\chi(\xi^*) = \frac{\xi^*}{\lambda_0}\{\sum \pi_k Q_k^2(\xi^*)\}^{-1} = \{\sum \pi_k^d (Q_k^d(\xi^*))^2 + \frac{\lambda_0}{\xi^*} Q_\infty(\xi^*) Q_\infty^d(\xi^*)\}^{-1}.$$

Since $Q_\infty(0) = 1$ and ξ^* lies between the first two zeros of $Q_\infty(x)$, we have $Q_\infty(\xi^*) < 0$. Further, $\xi^* \leqq \xi_1(\mathcal{P}^d)$, since $d\chi(x)$ is a solution of the moment problem associated with $\mathcal{P}^d$ (cf. Chihara (1978), Theorem II.4.4 (i)). Hence $Q_\infty^d(\xi^*) \geqq 0$. The maximal mass of $d\chi$ at ξ^* is $\{\sum \pi_k^d (Q_k^d(\xi^*))^2\}^{-1}$, thus we must have $Q_\infty^d(\xi^*) = 0$, i.e., $\xi^* = \xi_1(\mathcal{P}^d)$, and extremality of $d\chi$. Therefore, $d\chi = d\psi_{min}^d$.

ii. Let $\mu_0 = 0$ and

$$d\chi(x) = x d\psi_{min}(x)/\lambda_0 \ .$$

By similar arguments as those above one can show

$$d\chi(\xi_1(\mathcal{P})) = \left(\sum \pi_k^d (Q_k^d(\xi_1(\mathcal{P})))^2 + \frac{\lambda_0}{\xi_1(\mathcal{P})} Q_\infty(\xi_1(\mathcal{P})) Q_\infty^d(\xi_1(\mathcal{P}))\right)^{-1}.$$

Since $Q_\infty(\xi_1(\mathcal{P})) = 0$, $d\chi$ is extremal. Also $\mu_0^d \int d\chi(x)/x = \int d\psi_{min}$ $= 1$ ($d\psi_{min}$ having no mass at zero), so that $d\chi = d\psi_{max}^d$.

iii. Let $\mu_0 > 0$ and

$$d\chi(x) = \begin{cases} 0 & x = 0 \\ \mu_0 d\psi_{max}(x)/x & x > 0 \ . \end{cases}$$

Also, let ξ^* (> 0) denote the first spectral point of $d\psi_{max}$. Then, by a familiar argument,

$$d\chi(\xi^*) = \left(\sum \pi_k^d (Q_k^d(\xi^*))^2 - Q_\infty(\xi^*) Q_\infty^d(\xi^*)\right)^{-1} \ .$$

Clearly, $Q_\infty(\xi^*) > 0$ since $\xi^* > \xi_1(\mathcal{P})$, and $Q_\infty^d(\xi^*) \geqq 0$ since $\xi^* \leqq \xi_1(\mathcal{P}^d)$. On the other hand $Q_\infty(\xi^*) Q_\infty^d(\xi^*) \leqq 0$ since $\left(\sum \pi_k^d (Q_k^d(\xi^*))^2\right)^{-1}$ is the maximal mass $d\chi$ can have at ξ^*. It follows that $Q_\infty^d(\xi^*) = 0$, i.e., $\xi^* = \xi_1(\mathcal{P}^d)$, and $d\chi$ is extremal. Therefore, $d\chi = d\psi_{min}^d$.

iv. Let $\mu_0 > 0$ and

$$d\chi(x) = \begin{cases} 1 - \mu_0 \int_0^\infty d\psi_{min}(x)/x & x = 0 \\ \mu_0 d\psi_{min}(x)/x & x > 0 \ . \end{cases}$$

Then, by KM, Lemma 6 on p. 527, $d\chi(0) = (Q_\infty(0))^{-1}$. Hence, by (A.8) and (2.43)

$$d\chi(0) = \left(1+\mu_0 \sum_{k=0}^{\infty} (\lambda_k \pi_k)^{-1}\right)^{-1} = \left(\sum_{k=0}^{\infty} \pi_k^d\right)^{-1}.$$

Since $Q_k^d(0) = 1$ for all k, it follows by Lemma 2.3 that $d\chi$ is extremal. Finally, $d\chi = d\psi_{max}^d$ because it has mass at 0. □

REMARK. It is interesting to note that the relations between the distributions associated with minimal and maximal processes and their dual counterparts arise in recent work of Chihara (1982a), whose outlook is non-probabilistic.

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